Lecture 22. Eigenvectors and eigenvalues

Def Let A be an $n\times n$ matrix.

(1) A nonzero vector $\vec{v}\in\mathbb{R}^n$ with $A\vec{v}=\lambda\vec{v}$ for some $\lambda\in\mathbb{R}$ is called an eigenvector of A with eigenvalue λ.

e.g. $A=\begin{bmatrix}1&4\\2&3\end{bmatrix}$, $\vec{v}=\begin{bmatrix}1\\1\end{bmatrix}$

$\Rightarrow A\vec{v}=\begin{bmatrix}1&4\\2&3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}5\\5\end{bmatrix}=5\vec{v}$

$\Rightarrow \vec{v}$ is an eigenvector of A with eigenvalue 5.

(2) For an eigenvalue λ of A, the space $\text{Nul}(A-\lambda I)$ is called the λ-eigenspace of A.

Note While an eigenvector must be nonzero, an eigenvalue may be zero.

Thm A square matrix A has an eigenvalue $\lambda \iff \det(A-\lambda I)=0$

pf A has an eigenvalue λ

$\iff A\vec{v}=\lambda\vec{v}$ for some nonzero vector $\vec{v}\in\mathbb{R}^n$

$\iff (A-\lambda I)\vec{v}=\vec{0}$ for some nonzero vector $\vec{v}\in\mathbb{R}^n$

$\iff \text{Nul}(A-\lambda I)$ is not zero

$\iff \text{RREF}(A-\lambda I)$ has a column without a leading 1

$\iff \text{RREF}(A-\lambda I)\neq I$

$\iff \text{RREF}(A-\lambda I)$ is not invertible

$\iff \det(A-\lambda I)=0$

Note The λ-eigenspace $\text{Nul}(A-\lambda I)$ consists of a zero vector and the eigenvectors of A with eigenvalue λ.

Ex Consider the matrix

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

(1) Determine whether the vector

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

is an eigenvector of A.

Sol We have

$$A\vec{u} = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}$$

which is not a multiple of $\vec{u}$.

$\Rightarrow$ $\vec{u}$ is $\boxed{\text{not an eigenvector of } A}$

(2) Determine whether the vector

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

is an eigenvector of A.

Sol We have

$$A\vec{v} = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = \vec{v}$$

$\Rightarrow$ $\vec{v}$ is $\boxed{\text{an eigenvector of } A \text{ (with eigenvalue 1)}}$

(3) Determine whether $\lambda = 2$ is an eigenvalue of A

Sol We have

$$A - 2I = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix}$$

$\Rightarrow \det(A - 2I) = 0$

$\Rightarrow \lambda = 2$ is $\boxed{\text{an eigenvalue of } A}$

(4) Determine whether $\mu = 3$ is an eigenvalue of A

Sol We have

$$A - 3I = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -6 & -6 \\ -1 & 1 & 2 \\ 3 & -6 & -7 \end{bmatrix}$$

$\Rightarrow \det(A - 3I) = -2 \neq 0$

$\Rightarrow \mu = 3$ is $\boxed{\text{not an eigenvalue of } A}$

(5) Find an eigenvalue of A^3

Sol A has an eigenvalue $\lambda = 2$ as seen in (3).

Take an eigenvector $\vec{w}$ of A with eigenvalue $\lambda = 2$

$$\Rightarrow A^3\vec{w} = AA\underbrace{A\vec{w}}_{2\vec{w}} = AA(2\vec{w}) = 2A\underbrace{A\vec{w}}_{2\vec{w}} = 2A(2\vec{w}) = 2\cdot 2\underbrace{A\vec{w}}_{2\vec{w}} = \underbrace{2\cdot 2\cdot 2}_{2^3}\vec{w} = 8\vec{w}$$

$\Rightarrow \vec{w}$ is an eigenvector of A^3 with eigenvalue $\boxed{8}$

Note In general, A^m has an eigenvalue λ^m (with $\vec{w}$ as an eigenvector)

(6) Find a basis of the 2-eigenspace

Sol The 2-eigenspace is $\mathrm{Nul}(A-2I)$.

To find a basis, we solve the equation $(A-2I)\vec{x}=\vec{0}$

$$A-2I=\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \Rightarrow \mathrm{RREF}(A-2I)=\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_1-2x_2-2x_3=0 \Rightarrow x_1=2x_2+2x_3$$

Set $x_2=s$ and $x_3=t$ (free variables)

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2s+2t \\ s \\ t \end{bmatrix}=s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ A basis of $\mathrm{Nul}(A-2I)$ is given by $\boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}$

Note Any nonzero linear combination of these basis vectors yields an eigenvector of A with eigenvalue 2